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Research Article

New Integral Inequalities for Iterated Integrals with Applications

Ravi P. Agarwal, Cheon Seoung Ryoo, and Young-Ho Kim

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Some new nonlinear retarded integral inequalities of Gronwall type are established. These inequalities can be used as basic tools in the study of certain classes of integrodifferential equations.

1. Introduction

The Gronwall-type integral inequalities provide necessary tools in the study of the theory of differential equations, integral equations, and inequalities of various types. Some such inequalities can be found in the works of Agarwal, Deng et al. [1]. The result has been used in the study of global existence of solutions of a retarded differential equations and estimation of solution of function differential equation, Cheung [2]. The result has been used in the study of certain initial boundary value problem for hyperbolic partial differential equations, Cheung and Ma [3]. The result has been used in the study of global existence of solutions for a partial differential equations, Pachpatte [4–9]. The results have been applied in the study of certain properties of solutions for the integrodifferential equations, partial integrodifferential equations, retarded Volterra-Fredholm integral equations, retarded nonself–adjoint hyperbolic partial differential equations, Ye et al. [10]. The result has been used in the study of the Riemann-Liouville fractional integral equations, Zhao and Meng [11]. The result has been used in the study of integral equations. During the past few years, several authors (see [12–19] and some of the references cited therein) have established many other very useful Gronwall—like integral inequalities. Recently, in [16] a new interesting Gronwall—like integral inequality involving iterated integrals has been established.

Theorem 1.1. *Let $u(t)$ be nonnegative continuous function in $J=[\alpha,\beta]$ and let $a(t)$ be positive nondecreasing continuous function in J, and let $f_i(t,s)$, $i=1,\dots,n$, be nonnegative continuous functions for $\alpha\le s\le t\le\beta$ which are nondecreasing in t for fixed $s\in J$. If*

$$u(t)\le a(t)+\int_\alpha^t f_1(t,t_1)\left(\int_\alpha^{t_1} f_2(t_1,t_2)\cdots\left(\int_\alpha^{t_{n-1}} f_n(t_{n-1},t_n)u^p(t_n)dt_n\right)\cdots\right)dt_1 \quad (1.1)$$

for $t\in J$, where $p\ge 0$, $p\ne 1$, is a constant. Then $u(t)\le Y_1(t,t)$, where $Y_1(T,t)$ can be successively determined from the formulas

$$\begin{aligned} Y_n(T,t) &= \exp\left(\int_\alpha^t \sum_{i=1}^{n-1} f_i(T,s)ds\right)\\ &\quad\times\left[a^q(T)+q\int_\alpha^t f_n(T,s)\exp\left(-q\int_\alpha^s\sum_{i=1}^{n-1} f_i(T,\tau)d\tau\right)ds\right]^{1/q} \end{aligned} \quad (1.2)$$

for $t\in[\alpha,\beta_1)$, with $q=1-p$ and β_1 is chosen so that the expression between $[\cdots]$ is positive in the subinterval $[\alpha,\beta_1)$, and

$$\begin{aligned} Y_k(T,t) &= E_k(T,t)\left[a(T)+\int_\alpha^t f_k(T,s)\frac{Y_{k+1}(T,s)}{E_k(T,s)}ds\right],\\ E_k(T,t) &= \exp\left(\int_\alpha^t\left[\sum_{i=1}^{k-1} f_i(T,\tau)-f_k(T,\tau)\right]d\tau\right), \end{aligned} \quad (1.3)$$

for $k=n-1,\dots,1,\alpha\le t\le T\le\beta$.

The main aim of the present paper is to establish some nonlinear retarded inequalities, which extend the above theorem and other results appeared in [16]. We will also illustrate the usefulness of our results.

2. Gronwall-type inequalities

First we introduce some notation, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+=[0,\infty)$, $J=[\alpha,\beta]$ is the given subset of $\mathbb{R}$. Denote by $C^i(M,N)$ the class of all i-times continuously differentiable functions defined on the set M to the set N for $i=1,2,\dots$, and $C^0(M,N)=C(M,N)$.

Theorem 2.1. *Let $u(t)$ and $a(t)$ be nonnegative continuous functions in $J=[\alpha,\beta]$ with $a(t)$ nondecreasing in J, and let $f_i(t,s)$, $i=1,\dots,n$, be nonnegative continuous functions for $\alpha\le s\le t\le\beta$ which are nondecreasing in t for fixed $s\in J$. Suppose that $\phi\in C^1(J,J)$ is nondecreasing with $\phi(t)\le t$ on J, $g(u)$ is a nondecreasing continuous function for $u\in\mathbb{R}_+$ with $g(u)>0$ for $u>0$, and $\varphi\in C(\mathbb{R}_+,\mathbb{R}_+)$ is an increasing function with $\varphi(\infty)=\infty$. If*

$$\varphi(u(t))\le a(t)+\int_{\phi(\alpha)}^{\phi(t)} f_1(t,t_1)\left(\int_{\phi(\alpha)}^{\phi(t_1)} f_2(t_1,t_2)\cdots\left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} f_n(t_{n-1},t_n)g(u(t_n))dt_n\right)\cdots\right)dt_1 \quad (2.1)$$

for $t \in [\alpha,\beta]$, *then for* $t \in [\alpha, T_1]$,

$$u(t) \leq \varphi^{-1}\left[G^{-1}\left(G(a(t)) + \sum_{i=1}^{n}\int_{\phi(\alpha)}^{\phi(t)} f_i(t,s)ds\right)\right], \tag{2.2}$$

where

$$G(r) = \int_{r_0}^{r}\frac{ds}{s+g(s)}, \quad r \geq r_0 > 0, \tag{2.3}$$

G^{-1} *denotes the inverse function of* G, *and* $T_1 \in J$ *is chosen so that* $(G(a(t)) + \sum_{i=1}^{n}\int_{\phi(\alpha)}^{\phi(t)} f_i(t, s)ds) \in \mathrm{Dom}(G^{-1})$.

Proof. Let us first assume that $a(t) > 0$. Fix $T \in (\alpha,\beta]$. For $\alpha \leq t \leq T$, we obtain from (2.1)

$$\varphi(u(t)) \leq a(T) + \int_{\phi(\alpha)}^{\phi(t)} f_1(T,t_1)\left(\int_{\phi(\alpha)}^{\phi(t_1)} f_2(T,t_2)\cdots\left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} f_n(T,t_n)g(u(t_n))dt_n\right)\cdots\right)dt_1. \tag{2.4}$$

Now we introduce the functions

$$\begin{aligned} m_1(t) &= a(T) + \int_{\phi(\alpha)}^{\phi(t)} f_1(T,t_1)\left(\int_{\phi(\alpha)}^{\phi(t_1)} f_2(T,t_2)\cdots\left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} f_n(T,t_n)g(u(t_n))dt_n\right)\cdots\right)dt_1, \\ m_k(t) &= m_{k-1}(t) + \int_{\phi(\alpha)}^{\phi(t)} f_k(T,t_k)\left(\int_{\phi(\alpha)}^{\phi(t_k)} f_{k+1}(T,t_{k+1})\cdots\right. \\ &\qquad \left.\times\left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} f_n(T,t_n)g(m_{k-1}(t_n))dt_n\right)\cdots\right)dt_k, \end{aligned} \tag{2.5}$$

for $t \in [\alpha, T]$ and $k = 2,\ldots,n$. Then we have $m_k(\alpha) = a(T)$ for $k = 1,\ldots,n$, and $m_1(t) \leq m_2(t) \leq \cdots \leq m_n(t)$, $t \in [\alpha, T]$. From the inequality (2.4), we obtain $u(t) \leq \varphi^{-1}(m_1(t))$ or $u(t) \leq \varphi^{-1}(m_n(t))$, $t \in [\alpha, T]$. Moreover the function $m_1(t)$ is nondecreasing. Differentiating $m_1(t)$, we get

$$\begin{aligned} m_1'(t) &= f_1(T,\phi(t))\left[\int_{\phi(\alpha)}^{\phi(\phi(t))} f_2(T,t_2)\cdots\left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} f_n(T,t_n)g(u(t_n))dt_n)\cdots\right)dt_2\right]\phi'(t), \\ &\leq [-f_1(T,\phi(t))\phi'(t)m_1(t) + f_1(T,\phi(t))\phi'(t)m_2(t)]. \end{aligned} \tag{2.6}$$

Thus, induction with respect to k gives

$$m_k'(t) \leq \left(\sum_{i=1}^{k-1} f_i(T,\phi(t)) - f_k(T,\phi(t))\right)\phi'(t)m_k(t) + f_k(T,\phi(t))\phi'(t)m_{k+1}(t), \tag{2.7}$$

for $t \in [\alpha, T]$, $k = 1, 2, \ldots, n-1$. From the definition of the function $m_n(t)$ and inequality (2.7), we have

$$\begin{aligned}
m_n'(t) &= m_{n-1}'(t) + f_n(T, \phi(t)) g(m_{n-1}(\phi(t)) \phi'(t) \\
&\le \left[\left(\sum_{i=1}^{n-2} f_i(T, \phi(t)) \right) m_{n-1}(t) + f_{n-1}(T, \phi(t)) m_n(t) + f_n(T, \phi(t)) g(m_n(t)) \right] \phi'(t) \\
&\le \left[\left(\sum_{i=1}^{n-1} f_i(T, \phi(t)) \right) m_n(t) + f_n(T, \phi(t)) g(m_n(t)) \right] \phi'(t) \\
&\le \sum_{i=1}^{n} f_i(T, \phi(t)) \phi'(t) (m_n(t) + g(m_n(t))).
\end{aligned} \tag{2.8}$$

That is,

$$\frac{m_n'(t)}{m_n(t) + g(m_n(t))} \le \sum_{i=1}^{n} f_i(T, \phi(t)) \phi'(t). \tag{2.9}$$

Taking $t = s$ in (2.9) and then integrating it from α to any $t \in [\alpha, \beta]$, changing the variable and using the definition of the function G, we find

$$G(m_n(t)) \le G(m_n(\alpha)) + \sum_{i=1}^{n} \int_{\phi(\alpha)}^{\phi(t)} f_i(T, s) ds, \tag{2.10}$$

or

$$m_n(t) \le G^{-1}\left(G(m_n(\alpha)) + \sum_{i=1}^{n} \int_{\phi(\alpha)}^{\phi(t)} f_i(T, s) ds \right) \tag{2.11}$$

for $\alpha \le t \le T \le \beta$. Now, a combination of $u(t) \le \varphi^{-1}(m_n(t))$ and the last inequality gives the required inequality in (2.2) for $T = t$. If $a(t) = 0$, we replace $a(t)$ by some $\varepsilon > 0$ and subsequently let $\varepsilon \to 0$. This completes the proof. □

For the special case $g(u) = u^p$ ($p > 0$ is a constant), Theorem 2.1 gives the following retarded integral inequality for iterated integrals.

Corollary 2.2. *Let $u(t)$, $a(t)$, $f_i(t, s)$, $\phi(t)$, and $\varphi(u)$ be as in Theorem 2.1. And let $p > 0$ be a constant. Suppose that*

$$\varphi(u(t)) \le a(t) + \int_{\phi(\alpha)}^{\phi(t)} f_1(t, t_1) \left(\int_{\phi(\alpha)}^{\phi(t_1)} f_2(t_1, t_2) \cdots \left(\int_{\phi(\alpha)}^{\phi(t_{n-1})} f_n(t_{n-1}, t_n) u^p(t_n) dt_n \right) \cdots \right) dt_1 \tag{2.12}$$

for any $t \in [\alpha, \beta]$. Then, for any $t \in [\alpha, T_1]$,

$$u(t) \le \varphi^{-1}\left[G_1^{-1}\left(G_1(a(t)) + \sum_{i=1}^{n} \int_{\phi(\alpha)}^{\phi(t)} f_i(t, s) ds \right) \right], \tag{2.13}$$

where

$$G_1(r) = \int_{r_0}^{r} \frac{ds}{s+s^p}, \quad r \geq r_0 > 0, \tag{2.14}$$

G_1^{-1} *denotes the inverse function of* G_1, *and* $T_1 \in J$ *is chosen so that* $(G_1(a(t)) + \sum_{i=1}^{n} \int_{\phi(\alpha)}^{\phi(t)} f_i(t,s)ds) \in \mathrm{Dom}\,(G_1^{-1})$.

Remark 2.3. (i) When $\varphi(u) = u$ and $g(u) = u$, form Theorem 2.1, we derive the following retarded integral inequality:

$$u(t) \leq a(t) \exp\left(2\sum_{i=1}^{n} \int_{\phi(\alpha)}^{\phi(t)} f_i(t,s)ds\right). \tag{2.15}$$

(ii) When $\varphi(u) = u$, in Theorem 2.1, we obtain the following retarded integral inequality:

$$u(t) \leq G^{-1}\left(G(a(t)) + \sum_{i=1}^{n} \int_{\phi(\alpha)}^{\phi(t)} f_i(t,s)ds\right). \tag{2.16}$$

(iii) When $\varphi(u) = u^p$ ($p > 0$ is a constant) in Theorem 2.1, we have the following retarded integral inequality:

$$u(t) \leq \left[G^{-1}\left(G(a(t)) + \sum_{i=1}^{n} \int_{\phi(\alpha)}^{\phi(t)} f_i(t,s)ds\right)\right]^{1/p}. \tag{2.17}$$

Now we introduce the following notation. For $\alpha < \beta$, let $J_i = \{(t_1, t_2, \ldots, t_i) \in \mathbb{R}^i : \alpha \leq t_i \leq \cdots \leq t_1 \leq \beta\}$ for $i = 1, \ldots, n$.

Theorem 2.4. *Let $u(t)$ and $a(t)$ be nonnegative continuous functions in $J = [\alpha, \beta]$ with $a(t)$ nondecreasing in J, and let $p_i(t)$, $i = 1, \ldots, n$, be nonnegative continuous functions for $\alpha \leq t \leq \beta$. Suppose that $\phi \in C^1(J,J)$ is nondecreasing with $\phi(t) \leq t$ on J, $g(u)$ is a nondecreasing continuous function for $u \in \mathbb{R}_+$ with $g(u) > 0$ for $u > 0$, and $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing function with $\varphi(\infty) = \infty$. If*

$$\begin{aligned}
\varphi(u(t)) \leq{} & a(t) + \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) g(u(t_1)) dt_1 \\
& + \sum_{i=2}^{n} \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) \left(\int_{\phi(\alpha)}^{\phi(t_1)} p_2(t_2) \left(\cdots \left(\int_{\phi(\alpha)}^{\phi(t_{i-2})} p_{i-1}(t_{i-1}) \right.\right.\right. \\
& \left.\left.\left. \times \left(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i) g(u(t_i)) dt_i \right) dt_{i-1} \right) \cdots \right) dt_2 \right) dt_1,
\end{aligned} \tag{2.18}$$

for any $t \in J$, then

$$u(t) \leq \varphi^{-1}\left[G^{-1}(G(a(t)) + F(t))\right] \tag{2.19}$$

for $t \in [\alpha, T_2]$, *where* $T_2 \in I$ *is chosen so that* $(G(a(t)) + F(t)) \in \operatorname{Dom}(G^{-1})$,

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(\varphi^{-1}(s))}, \quad r \geq r_0 > 0, \tag{2.20}$$

G^{-1} *denotes the inverse function of* G, *and*

$$\begin{aligned} F(t) = {} & \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\,dt_1 + \sum_{i=2}^{n} \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\Bigg(\int_{\phi(\alpha)}^{\phi(t_1)} p_2(t_2)\Bigg(\cdots \Bigg(\int_{\phi(\alpha)}^{\phi(t_{i-2})} p_{i-1}(t_{i-1}) \\ & \times \Bigg(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i)\,dt_i \Bigg) dt_{i-1} \Bigg) \cdots \Bigg) dt_2 \Bigg) dt_1, \end{aligned} \tag{2.21}$$

for any $t \in I$.

Proof. Let the function $a(t)$ be positive. Define a function $v(t)$ by the right side of (2.18). Clearly, $v(t)$ is nondecreasing continuous, $u(t) \leq \varphi^{-1}(v(t))$ for $t \in I$ and $v(\alpha) = a(\alpha)$. Differentiating $v(t)$ and rewriting, we have

$$\frac{v'(t) - a'(t)}{\phi'(t) p_1(\phi(t))} - g(u(\phi(t))) \leq v_1(t), \tag{2.22}$$

where

$$\begin{aligned} v_1(t) = {} & \int_{\phi(\alpha)}^{\phi(t)} p_2(t_2) g(u(t_2))\,dt_2 \\ & + \sum_{i=3}^{n} \int_{\phi(\alpha)}^{\phi(t)} p_2(t_2)\Bigg(\int_{\phi(\alpha)}^{\phi(t_2)} p_3(t_3)\Bigg(\cdots \Bigg(\int_{\phi(\alpha)}^{\phi(t_{i-2})} p_{i-1}(t_{i-1}) \\ & \times \Bigg(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i) g(u(t_i))\,dt_i \Bigg) dt_{i-1} \Bigg) \cdots \Bigg) dt_3 \Bigg) dt_2. \end{aligned} \tag{2.23}$$

Now differentiating $v_1(t)$ and rewriting, we get

$$\frac{v_1'(t)}{\phi'(t) p_2(\phi(t))} - g(u(\phi(t))) \leq v_2(t), \tag{2.24}$$

where

$$\begin{aligned} v_2(t) = {} & \int_{\phi(\alpha)}^{\phi(t)} p_3(t_3) g(u(t_3))\,dt_3 \\ & + \sum_{i=4}^{n} \int_{\phi(\alpha)}^{\phi(t)} p_3(t_3)\Bigg(\int_{\phi(\alpha)}^{\phi(t_3)} p_4(t_4)\Bigg(\cdots \Bigg(\int_{\phi(\alpha)}^{\phi(t_{i-2})} p_{i-1}(t_{i-1}) \\ & \times \Bigg(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i) g(u(t_i))\,dt_i \Bigg) dt_{i-1} \Bigg) \cdots \Bigg) dt_4 \Bigg) dt_3. \end{aligned} \tag{2.25}$$

Continuing in this way, we obtain

$$\frac{v'_{n-2}(t)}{\phi'(t)p_{n-1}(\phi(t))} - g(u(\phi(t))) \le v_{n-1}(t), \tag{2.26}$$

where

$$v_{n-1}(t) = \int_{\phi(\alpha)}^{\phi(t)} p_n(t_n)g(u(t_n))\,dt_n. \tag{2.27}$$

From the definition of $v_{n-1}(t)$ and the inequality $u(t) \le \varphi^{-1}(v(t))$, we find

$$\frac{v'_{n-1}(t)}{g(\varphi^{-1}(v(t)))} \le \phi'(t)p_n(\phi(t)). \tag{2.28}$$

Integrating the inequality (2.28), we get

$$\int_\alpha^t \frac{v'_{n-1}(s)}{g(\varphi^{-1}(v(s)))}ds \le \int_{\phi(\alpha)}^{\phi(t)} p_n(s)ds. \tag{2.29}$$

Now integrating by parts the left–hand side of (2.29), we obtain

$$\begin{aligned}\int_\alpha^t \frac{v'_{n-1}(s)}{g(\varphi^{-1}(v(s)))}ds &= \frac{v_{n-1}(t)}{g(\varphi^{-1}(v(t)))} + \int_\alpha^t \frac{v_{n-1}g'(\varphi^{-1}(v))}{g^2(\varphi^{-1}(v))}\frac{v'}{\varphi'[\varphi^{-1}(v)]}ds \\ &\ge \frac{v_{n-1}(t)}{g(\varphi^{-1}(v(t)))}.\end{aligned} \tag{2.30}$$

From the inequalities (2.29) and (2.30), we have

$$\frac{v_{n-1}(t)}{g(\varphi^{-1}(v(t)))} \le \int_{\phi(\alpha)}^{\phi(t)} p_n(s)ds. \tag{2.31}$$

Next from the inequality (2.26), we observe that

$$v'_{n-2}(t) \le \phi'(t)p_{n-1}(\phi(t))g(u(\phi(t))) + \phi'(t)p_{n-1}(\phi(t))v_{n-1}(t), \tag{2.32}$$

Thus, it follows that

$$\begin{aligned}\frac{v'_{n-2}(t)}{g(\varphi^{-1}(v(t)))} &\le \phi'(t)p_{n-1}(\phi(t))\frac{g(u(\phi(t)))}{g(\varphi^{-1}(v(t)))} + \phi'(t)p_{n-1}(\phi(t))\frac{v_{n-1}(t)}{g(\varphi^{-1}(v(t)))} \\ &\le \phi'(t)p_{n-1}(\phi(t)) + \phi'(t)p_{n-1}(\phi(t))\frac{v_{n-1}(t)}{g(\varphi^{-1}(v(t)))}.\end{aligned} \tag{2.33}$$

Using the same procedure from (2.29) to (2.31) to the inequality (2.33), we get

$$\frac{v_{n-2}(t)}{g(\varphi^{-1}(v(t)))} \le \int_{\phi(\alpha)}^{\phi(t)} p_{n-1}(t_1)dt_1 + \int_{\phi(\alpha)}^{\phi(t)} p_{n-1}(t_1)\frac{v_{n-1}(t_1)}{g(\varphi^{-1}(v(t_1)))}dt_1. \tag{2.34}$$

Now combining the inequalities (2.31) and (2.34), we find

$$\frac{v_{n-2}(t)}{g(\varphi^{-1}(v(t)))} \le \int_{\phi(\alpha)}^{\phi(t)} p_{n-1}(t_1)dt_1 + \int_{\phi(\alpha)}^{\phi(t)} p_{n-1}(t_1)\int_{\phi(\alpha)}^{\phi(t_1)} p_n(t_2)dt_2dt_1. \tag{2.35}$$

Proceeding in this way we arrive at

$$\begin{aligned}\frac{v_1(t)}{g(\varphi^{-1}(v(t)))} \le &\int_{\phi(\alpha)}^{\phi(t)} p_2(t_1)dt_1 + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)} p_2(t_1)\left(\int_{\phi(\alpha)}^{\phi(t_1)} p_3(t_2)\left(\cdots\int_{\phi(\alpha)}^{\phi(t_{n-2})} p_n(t_{n-1})dt_{n-1}\cdots\right)dt_2\right)dt_1.\end{aligned} \tag{2.36}$$

On the other hand, from the inequality (2.22), we have

$$v'(t) - a'(t) \le \phi'(t)p_1(\phi(t))g(u(\phi(t))) + \phi'(t)p_1(\phi(t))v_1(t), \tag{2.37}$$

or

$$\begin{aligned}\frac{v'(t)-a'(t)}{g(\varphi^{-1}(v(t)))} &\le \phi'(t)p_1(\phi(t))\frac{g(u(\phi(t)))}{g(\varphi^{-1}(v(t)))} + \phi'(t)p_1(\phi(t))\frac{v_1(t)}{g(\varphi^{-1}(v(t)))} \\ &\le \phi'(t)p_1(\phi(t)) + \phi'(t)p_1(\phi(t))\frac{v_1(t)}{g(\varphi^{-1}(v(t)))},\end{aligned} \tag{2.38}$$

that is,

$$\frac{v'(t)}{g(\varphi^{-1}(v(t)))} - \frac{a'(t)}{g(\varphi^{-1}(a(t)))} \le \phi'(t)p_1(\phi(t)) + \phi'(t)p_1(\phi(t))\frac{v_1(t)}{g(\varphi^{-1}(v(t)))}. \tag{2.39}$$

Setting $t = t_1$ and integrating from α to t, and using the definition of G, we obtain

$$G(v(t)) \le G(a(t)) + \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)dt_1 + \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\frac{v_1(t_1)}{g(\varphi^{-1}(v(t_1)))}dt_1. \tag{2.40}$$

Consequently, using (2.36) to the inequality (2.40), we get

$$v(t) \le G^{-1}[G(a(t)) + F(t)], \tag{2.41}$$

where the function $F(t)$ is defined in (2.21). Now, the desired inequality in (2.24) follows by the inequality $u(t) \le \varphi^{-1}(v(t))$. If $a(t) = 0$, we replace $a(t)$ by some $\varepsilon > 0$ and subsequently let $\varepsilon \to 0$. This completes the proof. □

For the special case $\varphi(u) = u^p$ ($p > 1$ is a constant), Theorem 2.4 gives the following retarded integral inequality for iterated integrals.

Corollary 2.5. *Let $u(t)$, $a(t)$, $p_i(t)$, $\phi(t)$, and $g(u)$ be as in Theorem 2.4. And let $p > 0$ be a constant. If*

$$\begin{aligned} u^p(t) \le a(t) &+ \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) g(u(t_1))\,dt_1 \\ &+ \sum_{i=2}^{n} \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\left(\int_{\phi(\alpha)}^{\phi(t_1)} p_2(t_2)\left(\cdots\left(\int_{\phi(\alpha)}^{\phi(t_{i-2})} p_{i-1}(t_{i-1})\right.\right.\right. \\ &\left.\left.\left. \times\left(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i) g(u(t_i))\,dt_i\right)dt_{i-1}\right)\cdots\right)dt_2\right)dt_1, \end{aligned} \tag{2.42}$$

for any $t \in J$, then

$$u(t) \le \left[G^{-1}(G(a(t)) + F(t))\right]^{1/p} \tag{2.43}$$

for $t \in [\alpha, T_3]$, where $T_3 \in I$ is chosen so that $(G_1(a(t)) + F(t)) \in \mathrm{Dom}\,(G_1^{-1})$,

$$G_1(r) = \int_{r_0}^{r} \frac{ds}{g(v^{1/p}(s))}, \quad r \ge r_0 > 0, \tag{2.44}$$

G^{-1} denotes the inverse function of G, and the function $F(t)$ is defined in (2.21) for any $t \in I$.

Theorem 2.6. *Let $u(t)$ and $a(t)$ be nonnegative continuous functions in $J = [\alpha,\beta]$ with $a(t)$ nondecreasing in J, and let $f_i(t)$ and $p_i(t)$, $i = 1,\dots,n$, be nonnegative continuous functions for $\alpha \le t \le \beta$. Suppose that $\phi \in C^1(J,J)$ is nondecreasing with $\phi(t) \le t$ on J, $g(u)$ is a nondecreasing continuous function for $u \in \mathbb{R}_+$ with $g(u) > 0$ for $u > 0$, and $\varphi \in C(\mathbb{R}_+,\mathbb{R}_+)$ is an increasing function with $\varphi(\infty) = \infty$. If*

$$\begin{aligned} \varphi(u(t)) \le a(t) &+ \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) f_1(t_1) u(t_1) g(u(t_1))\,dt_1 \\ &+ \sum_{i=2}^{n} \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\left(\int_{\phi(\alpha)}^{\phi(t_1)} p_2(t_2)\left(\cdots\left(\int_{\phi(\alpha)}^{\phi(t_{i-2})} p_{i-1}(t_{i-1})\right.\right.\right. \\ &\left.\left.\left. \times\left(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i) f_i(t_i) u(t_i) g(u(t_i))\,dt_i\right)dt_{i-1}\right)\cdots\right)dt_2\right)dt_1, \end{aligned} \tag{2.45}$$

for any $t \in J$, then

$$u(t) \le \varphi^{-1}\left\{\Phi^{-1}\left[G_2^{-1}\left(G_2[\Phi(a(t))] + F_1(t)\right)\right]\right\} \tag{2.46}$$

for $t \in [\alpha, T_4]$, *where* $T_4 \in I$ *is chosen so that* $(G_2[\Phi(a(t))] + F_1(t)) \in \operatorname{Dom}(G_2^{-1})$, $[G_2^{-1}(G_2[\Phi(a(t))] + F_1(t))] \in \operatorname{Dom}(\Phi^{-1})$,

$$G_2(r) = \int_{r_0}^{r} \frac{ds}{g(\varphi^{-1}(\Phi^{-1}(s)))}, \quad r \geq r_0 > 0,$$
$$\Phi(r) = \int_{r_0}^{r} \frac{ds}{\varphi^{-1}(s)}, \quad r \geq r_0 > 0, \tag{2.47}$$

G_2^{-1} *denotes the inverse function of* G_2, *and*

$$\begin{aligned} F_1(t) = {} & \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) f_1(t) dt_1 \\ & + \sum_{i=2}^{n} \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) \left(\int_{\phi(\alpha)}^{\phi(t_1)} p_2(t_2) \times \left(\cdots \left(\int_{\phi(\alpha)}^{\phi(t_{i-2})} p_{i-1}(t_{i-1}) \right.\right.\right. \\ & \left.\left.\left. \times \left(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i) f_i(t_i) dt_i \right) dt_{i-1} \right) \cdots \right) dt_2 \right) dt_1, \end{aligned} \tag{2.48}$$

for any $t \in I$.

Proof. Let the function $a(t)$ be positive. Define a function $w(t)$ by the right side of (2.45). Clearly, $w(t)$ is a nondecreasing continuous function, $u(t) \leq \varphi^{-1}(w(t))$ for $t \in I$ and $w(\alpha) = a(\alpha)$. Differentiating $w(t)$ and rewriting, we have

$$\frac{w'(t) - a'(t)}{\phi'(t) p_1(\phi(t))} - f_1(\phi(t)) u(\phi(t)) g(u(\phi(t))) \leq w_1(t), \tag{2.49}$$

where

$$\begin{aligned} w_1(t) = {} & \int_{\phi(\alpha)}^{\phi(t)} p_2(t_2) f_2(t_2) u(t_2) g(u(t_2)) dt_2 \\ & + \sum_{i=3}^{n} \int_{\phi(\alpha)}^{\phi(t)} p_2(t_2) \left(\int_{\phi(\alpha)}^{\phi(t_2)} p_3(t_3) \left(\cdots \left(\int_{\phi(\alpha)}^{\phi(t_{i-2})} p_{i-1}(t_{i-1}) \right.\right.\right. \\ & \left.\left.\left. \times \left(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i) f_i(t_i) u(t_i) g(u(t_i)) dt_i \right) dt_{i-1} \right) \cdots \right) dt_3 \right) dt_2. \end{aligned} \tag{2.50}$$

Now differentiating $v_1(t)$ and rewriting, we get

$$\frac{w_1'(t)}{\phi'(t) p_2(\phi(t))} - f_2(\phi(t)) u(\phi(t)) g(u(\phi(t))) \leq w_2(t), \tag{2.51}$$

where

$$\begin{aligned}w_2(t) &= \int_{\phi(\alpha)}^{\phi(t)} p_3(t_3) f_3(t_3) u(t_3) g(u(t_3))\,dt_3\\&\quad+\sum_{i=4}^{n}\int_{\phi(\alpha)}^{\phi(t)} p_3(t_3)\left(\int_{\phi(\alpha)}^{\phi(t_3)} p_4(t_4)\left(\cdots\left(\int_{\phi(\alpha)}^{\phi(t_{i-2})} p_{i-1}(t_{i-1})\right.\right.\right.\\&\qquad\left.\left.\left.\times\left(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i) f_i(t_i) u(t_i) g(u(t_i))\,dt_i\right)dt_{i-1}\right)\cdots\right)dt_4\right)dt_3.\end{aligned} \tag{2.52}$$

Continuing in this way, we obtain

$$\frac{w'_{n-2}(t)}{\phi'(t)p_{n-1}(\phi(t))} - f_{n-1}(\phi(t))u(\phi(t))g(u(\phi(t))) \le w_{n-1}(t), \tag{2.53}$$

where

$$w_{n-1}(t) = \int_{\phi(\alpha)}^{\phi(t)} p_n(t_n) f_n(t_n) u(t_n) g(u(t_n))\,dt_n. \tag{2.54}$$

From the definition of $w_{n-1}(t)$ and the inequality $u(t) \le \varphi^{-1}(w(t))$, we get

$$\frac{w'_{n-1}(t)}{\varphi^{-1}(w(t))} \le \phi'(t)p_n(\phi(t)) f_n(\phi(t)) g(\varphi^{-1}(w(\phi(t)))). \tag{2.55}$$

Integrating the inequality (2.55), we have

$$\int_\alpha^t \frac{w'_{n-1}(s)}{\varphi^{-1}(w(s))}ds \le \int_{\phi(\alpha)}^{\phi(t)} p_n(s) f_n(s) g(\varphi^{-1}(w(s)))\,ds. \tag{2.56}$$

Next integrating by parts the left–hand side of (2.56), we obtain

$$\int_\alpha^t \frac{w'_{n-1}(s)}{\varphi^{-1}(w(s))}ds = \frac{w_{n-1}(t)}{\varphi^{-1}(w(t))} + \int_\alpha^t \frac{w_{n-1}}{(\varphi^{-1}(w))^2}\frac{w'}{\varphi'[\varphi^{-1}(w)]}ds \ \ge \frac{w_{n-1}(t)}{\varphi^{-1}(w(t))}. \tag{2.57}$$

From the inequalities (2.56) and (2.57), we get

$$\frac{w_{n-1}(t)}{\varphi^{-1}(w(t))} \le \int_{\phi(\alpha)}^{\phi(t)} p_n(s) f_n(s) g(\varphi^{-1}(w(s)))\,ds. \tag{2.58}$$

Now from the inequality (2.53), we observe that

$$\begin{aligned}w'_{n-2}(t) &\le \phi'(t)p_{n-1}(\phi(t))w_{n-1}(t)\\&\quad+\phi'(t)p_{n-1}(\phi(t)) f_{n-1}(\phi(t))\varphi^{-1}(w(t))g(\varphi^{-1}(w(t))).\end{aligned} \tag{2.59}$$

Also, from the inequality (2.59), we have

$$\frac{w'_{n-2}(t)}{\varphi^{-1}(w(t))} \leq \phi'(t)p_{n-1}(\phi(t))\frac{w_{n-1}(t)}{\varphi^{-1}(w(t))} + \phi'(t)p_{n-1}(\phi(t))f_{n-1}(\phi(t))g(\varphi^{-1}(w(t))). \tag{2.60}$$

Using the same procedure from (2.56) to (2.58) to the inequality (2.60), we find

$$\frac{w_{n-2}(t)}{\varphi^{-1}(w(t))} \leq \int_{\phi(\alpha)}^{\phi(t)} p_{n-1}(t_1)\frac{w_{n-1}(t_1)}{\varphi^{-1}(w(t_1))}dt_1 + \int_{\phi(\alpha)}^{\phi(t)} p_{n-1}(t_1)f_{n-1}(t_1)g(\varphi^{-1}(w(t_1)))dt_1. \tag{2.61}$$

Next using (2.58) in the inequality (2.61), we get

$$\frac{w_{n-2}(t)}{\varphi^{-1}(w(t))} \leq \int_{\phi(\alpha)}^{\phi(t)} p_{n-1}(t_1)\int_{\phi(\alpha)}^{\phi(t_1)} p_n(s)f_n(s)g(\varphi^{-1}(w(s)))ds\,dt_1 + \int_{\phi(\alpha)}^{\phi(t)} p_{n-1}(t_1)f_{n-1}(t_1)g(\varphi^{-1}(w(t_1)))dt_1. \tag{2.62}$$

Proceeding in this way, we arrive at

$$\frac{w_1(t)}{\varphi^{-1}(w(t))} = \int_{\phi(\alpha)}^{\phi(t)} p_2(t_1)f_2(t_1)g(\varphi^{-1}(w(t_1)))dt_1 + \cdots + \int_{\phi(\alpha)}^{\phi(t)} p_2(t_1)\left(\cdots\int_{\phi(\alpha)}^{\phi(t_{n-2})} p_n(t_s)f_n(t_s)g(\varphi^{-1}(w(t_s)))ds\cdots\right)dt_1. \tag{2.63}$$

On the other hand, from the inequality (2.49), we have

$$w'(t)-a'(t) \leq \phi'(t)p_1(\phi(t))f_1(\phi(t))\varphi^{-1}(w(t))g(\varphi^{-1}(w(t))) + \phi'(t)p_1(\phi(t))w_1(t), \tag{2.64}$$

or

$$\frac{w'(t)-a'(t)}{\varphi^{-1}(w(t))} \leq \phi'(t)p_1(\phi(t))\frac{w_1(t)}{\varphi^{-1}(w(t))} + \phi'(t)p_1(\phi(t))f_1(\phi(t))g(\varphi^{-1}(w(t))). \tag{2.65}$$

Now, theleft-hand side of the inequality (2.65) implies that

$$\frac{w'(t)}{\varphi^{-1}(w(t))} - \frac{a'(t)}{\varphi^{-1}(a(t))} \leq \frac{w'(t)-a'(t)}{\varphi^{-1}(w(t))}. \tag{2.66}$$

In the inequalities (2.65) and (2.66), setting $t = t_1$, integrating from α to t, and using the definition of Φ, we obtain

$$\Phi(w(t)) \leq \Phi(a(t)) + \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\frac{w_1(t_1)}{\varphi^{-1}(w(t_1))}dt_1 + \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)f_1(t_1)g(\varphi^{-1}(w(t)))dt_1. \tag{2.67}$$

Consequently, from the inequalities (2.63) and (2.67), we get

$$w(t) \leq \Phi^{-1}[k(t)], \tag{2.68}$$

where the function $k(t)$ is defined by

$$\begin{aligned} k(t) = \Phi(a(T)) &+ \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)f_1(t_1)g(\varphi^{-1}(w(t_1)))dt_1 \\ &+ \sum_{i=2}^{n}\int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\left(\int_{\phi(\alpha)}^{\phi(t_1)} p_2(t_2)\right. \\ &\left.\times\left(\cdots\left(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i)f_i(t_i)g(\varphi^{-1}(w(t_i)))dt_i\right)\cdots\right)dt_2\right)dt_1, \end{aligned} \tag{2.69}$$

for some fixed T, $t \leq T \leq \beta$. Clearly, $k(t)$ is a nondecreasing continuous function and $k(\alpha) = \Phi(a(T))$. Differentiating $k(t)$ and rewriting, we have

$$\frac{k'(t)}{\phi'(t)p_1(\phi(t))} - f_1(\phi(t))g(\varphi^{-1}(w(\phi(t)))) \leq k_1(t), \tag{2.70}$$

where

$$\begin{aligned} k_1(t) = &\int_{\phi(\alpha)}^{\phi(t)} p_2(t_2)f_2(t_2)g(\varphi^{-1}(w(t_2)))dt_2 \\ &+ \sum_{i=3}^{n}\int_{\phi(\alpha)}^{\phi(t)} p_2(t_2)\left(\int_{\phi(\alpha)}^{\phi(t_2)} p_3(t_3)\left(\cdots\left(\int_{\phi(\alpha)}^{\phi(t_{i-2})} p_{i-1}(t_{i-1})\right.\right.\right. \\ &\left.\left.\left.\times\left(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i)f_i(t_i)g(\varphi^{-1}(w(t_i)))dt_i\right)dt_{i-1}\right)\cdots\right)dt_3\right)dt_2. \end{aligned} \tag{2.71}$$

Using the same procedures from (2.51) to (2.65) to the equality (2.71), we have

$$\frac{k_1(t)}{g(\varphi^{-1}(w(t)))} \le \int_{\phi(\alpha)}^{\phi(t)} p_2(t_2) f_2(t_2) dt_2 + \sum_{i=3}^{n} \int_{\phi(\alpha)}^{\phi(t)} p_2(t_2) \left(\int_{\phi(\alpha)}^{\phi(t_2)} p_3(t_3) \cdots \left(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i) f_i(t_i) dt_i \right) \cdots dt_3 \right) dt_2, \tag{2.72}$$

$$\frac{k'(t)}{g(\varphi^{-1}(\Phi^{-1}(k(t))))} \le \phi'(t) p_1(\phi(t)) \frac{k_1(t)}{\varphi^{-1}(w(t))} + \phi'(t) p_1(\phi(t)) f_1(\phi(t)). \tag{2.73}$$

In the inequality (2.73), setting $t = s$ and integrating from α to t, using the definition of G_2, we obtain

$$G_2(k(t)) \le G_2(k(\alpha)) + \int_{\phi(\alpha)}^{\phi(t)} p_1(s) \frac{k_1(s)}{g(\varphi^{-1}(w(s)))} ds + \int_{\phi(\alpha)}^{\phi(t)} p_1(s) f_1(s) ds. \tag{2.74}$$

Finally, from the inequalities (2.72) and (2.74), we get

$$k(t) \le G_2^{-1}[G_2(\Phi(a(T))) + F_1(t)], \tag{2.75}$$

where the function $F_1(t)$ is defined in (2.46). In particular, for $T = t$, we find that the desired inequality (2.46) follows by the inequalities $u(t) \le \varphi^{-1}(w(t))$ and $w(t) \le \Phi^{-1}(k(t))$. This completes the proof. □

When $\varphi(u) = u^p$ ($p > 1$ is a constant) in Theorem 2.6, we get the following Ou-Iang type-retarded integral inequality with iterated integrals.

Corollary 2.7. *Let u, a, f_i, p_i, ϕ, and $g(u)$ be as in Theorem 2.6 and let $p > 1$ be a constant. If*

$$u^p(t) \le a(t) + \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) f_1(t_1) u(t_1) g(u(t_1)) dt_1 + \sum_{i=2}^{n} \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) \left(\int_{\phi(\alpha)}^{\phi(t_1)} p_2(t_2) \left(\cdots \left(\int_{\phi(\alpha)}^{\phi(t_{i-2})} p_{i-1}(t_{i-1}) \times \left(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i) f_i(t_i) u(t_i) g(u(t_i)) dt_i \right) dt_{i-1} \right) \cdots \right) dt_2 \right) dt_1, \tag{2.76}$$

for any $t \in J$, then

$$u(t) \le \left\{ G_3^{-1}\left[G_3(a^{(p-1)/p}(t)) + \frac{p-1}{p} F_1(t) \right] \right\}^{1/(p-1)} \tag{2.77}$$

for $t\in[\alpha,T_3]$, *where* $T_3\in I$ *is chosen so that* $[G_3(a^{(p-1)/p}(t))+((p-1)/p)F_1(t)]\in\mathrm{Dom}\,(G_3^{-1})$,

$$G_3(r)=\int_{r_0}^{r}\frac{ds}{g(v^{1/(p-1)}(s))},\quad r\geq r_0>0, \tag{2.78}$$

G_3^{-1} *denotes the inverse function of* G_3, *and* $F_1(t)$ *is defined in (2.48) for any* $t\in I$.

3. Applications

In this section, we will show that our results are directly useful in proving the global existence of solutions to certain integrodifferential equations. First we consider the following integrodifferential equation:

$$px^{p-1}(t)x'(t)=F\left(t,x(t-\tau(t)),\int_{\alpha}^{t-\tau(t)}G(t_1,x(t_1-\tau(t_1)))\,dt_1\right) \tag{3.1}$$

for $t\in I$, where $p>1$ is constant, let $F\in C(I\times\mathbb{R}^2,\mathbb{R})$, $G\in C^1(I\times\mathbb{R},\mathbb{R})$, and $\tau\in C^1(I,I)$ be nonincreasing with $t-\tau(t)\geq 0$, $t-\tau(t)\in C^1(I,I)$, $\tau'(t)<1$, and $\tau(\alpha)=0$. The following result provides a bound on the solutions of (3.1).

Theorem 3.1. *Assume that* $F:I\times\mathbb{R}^2\to\mathbb{R}$ *is a continuous function, and there exist continuous nonnegative functions* $b_i(t)$, $i=1,2$, *such that*

$$\begin{aligned}|F(t,u,v)|&\leq b_1(t)g(|u|)+b_1(t)|v|,\\ |G(s,w)|&\leq b_2(s)g(|w|),\end{aligned} \tag{3.2}$$

where the function g *is the same as in Theorem 2.4. Let* $M=\max_{x\in I}(1/(1-\tau'(x)))$. *If* $x(\eta)$ *is any solution of the problem (3.1), then*

$$|x(\eta)|\leq\left[G_1^{-1}(G_1(|x(\alpha)|^p)+B[Mb_1(\overline{\eta_1}),M^2b_1(\overline{\eta_1})b_2(\overline{\eta_2})])\right]^{1/p}, \tag{3.3}$$

where the functions G_1,G_1^{-1} *are as in Corollary 2.5,* $\overline{\eta_1}=\eta_1+\tau(t_1)$, $\overline{\eta_2}=\eta_2+\tau(t_2)$, *for* $\eta_1,\eta_2\in I$, *and*

$$B[Mb_1(\overline{\eta_1}),M^2b_1(\overline{\eta_1})b_2(\overline{\eta_2})]=\int_{\phi(\alpha)}^{\phi(\eta)}Mb_1(\overline{\eta_1})\,d\eta_1+\int_{\phi(\alpha)}^{\phi(\eta)}\int_{\phi(\alpha)}^{\phi(\eta_1)}M^2b_2(\overline{\eta_1})b_3(\overline{\eta_2})\,d\eta_2\,d\eta_1, \tag{3.4}$$

where $\phi(\gamma)=\gamma-\tau(\gamma)$ *for* $\gamma\in I$.

Proof. It is easy to see that the solution $x(\eta)$ of the problem (3.1) satisfies the equivalent integral equation

$$x^p(\eta)=x^p(\alpha)+\int_{\alpha}^{\eta}F\left(t_1,x(t_1)-\tau(t_1),\int_{\alpha}^{t_1-\tau(t_1)}G(t_2,x(t_2-\tau(t_2)))\,dt_2\right)dt_1. \tag{3.5}$$

From (3.2), and making the change of variables, we have

$$\begin{aligned}
|x(\eta)|^{p} &\le |x(\alpha)|^{p} + \int_{\alpha}^{\eta} b_1(t_1)g(|x(t_1-\tau(t_1))|)dt_1 \\
&\quad + \int_{\alpha}^{\eta}\int_{\alpha}^{t_1-\tau(t_1)} b_1(t_1)b_2(t_2)g(|x(t_2-\tau(t_2))|)dt_2\,dt_1 \\
&\le |x(\alpha)|^{p} + \int_{\alpha}^{\eta-\tau(\eta)} Mb_1(\overline{\eta_1})g(|x(\eta_1)|)d\eta_1 \\
&\quad + \int_{\alpha}^{\eta-\tau(\eta)}\int_{\alpha}^{\eta_1-\tau(\eta_1)} M^2 b_1(\overline{\eta_1})b_2(\overline{\eta_2})g(|x(\eta_2)|)d\eta_2 d\eta_1,
\end{aligned} \tag{3.6}$$

where $\overline{\eta_1}=\eta_1+\tau(t_1)$, $\overline{\eta_2}=\eta_2+\tau(t_2)$, for $\eta_1,\eta_2\in I$. Now an immediate application of the inequality established in Corollary 2.5 to (3.6) yields the desired result. □

We next consider the following integrodifferential equation:

$$(h(t)x'(t))' = F\left(t, x(t-\tau(t)), \int_{\alpha}^{t} G(t_1, x(t_1-\tau(t_1)))dt_1\right) \tag{3.7}$$

for $t\in I$, $F\in C(I\times\mathbb{R}^2,\mathbb{R})$, $G\in C^1(I\times\mathbb{R},\mathbb{R})$, and h is positive and continuous in I. The following theorem provides an upper bound on the solutions of (3.7).

Theorem 3.2. *Assume that $F: I\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function, and there exist continuous nonnegative functions $f_i(t)$, $i=2,3$, such that*

$$\begin{aligned}
|F(t,u,v)| &\le f_2(t)[|u|g(|u|)+|v|], \\
|G(s,w)| &\le f_3(s)|w|g(|w|),
\end{aligned} \tag{3.8}$$

where function g is the same as in Theorem 2.6. If $x(t)$ is any solution of the problem (3.7), then

$$|x(t)| \le \exp\left[G_e^{-1}(G_e(\ln a)+C[f_2(\overline{s_2}), f_2(\overline{s_2})f_3(\overline{s_3})])\right]-1, \tag{3.9}$$

where $G_e(r)=\int_{r_0}^{r}(ds/g(e^s))$ for $r\ge r_0>0$, $a=1+|x(\alpha)|+h(\alpha)|x'(\alpha)|\int_{\phi(\alpha)}^{\phi(t)}\frac{1}{h(\overline{s_1})}ds_1$, and

$$\begin{aligned}
C[f_2, f_2f_3] &= \int_{\phi(\alpha)}^{\phi(t)}\frac{1}{h(\overline{s_1})}\int_{\phi(\alpha)}^{\phi(t_1)} f_2(\overline{s_2})M^2 ds_2 ds_1 \\
&\quad + \int_{\phi(\alpha)}^{\phi(t)}\frac{1}{h(\overline{s_1})}\int_{\phi(\alpha)}^{\phi(t_1)} f_2(\overline{s_2})\int_{\phi(\alpha)}^{\phi(t_2)} f_3(\overline{s_3})M^3 ds_3 ds_2 ds_1,
\end{aligned} \tag{3.10}$$

$\overline{s_1}=s_1+\tau(t_1)$, $\overline{s_2}=s_2+\tau(t_2)$, and $\overline{s_3}=s_3+\tau(t_2)$ for $s_1,s_2,s_3\in I$.

Proof. It is easy to see that the solution $x(t)$ of the problem (3.7) satisfies the equivalent integral equation

$$\begin{aligned} x(t) = {} & x(\alpha) + h(\alpha)x'(\alpha)\int_\alpha^t \frac{1}{h(t_1)}dt_1 \\ & + \int_\alpha^t \frac{1}{h(t_1)}\int_\alpha^{t_1} F\left(t_2, x(t_2-\tau(t_2)),\ \int_\alpha^{t_2} G(t_3), x(t_3-\tau(t_3))dt_3\right)dt_2 dt_1. \end{aligned} \tag{3.11}$$

From (3.8), and making the change of variables, we have

$$\begin{aligned} |x(t)| + 1 \le {} & 1 + x(\alpha) + h(\alpha)|x'(\alpha)|\int_{\phi(\alpha)}^{\phi(t)} \frac{1}{h(\overline{s_1})}ds_1 \\ & + \int_{\phi(\alpha)}^{\phi(t)} \frac{1}{h(\overline{s_1})}\int_{\phi(\alpha)}^{\phi(t_1)} f_2(\overline{s_2})M^2|x(s_2)|g(|x(s_2)|)ds_2 ds_1 \\ & + \int_{\phi(\alpha)}^{\phi(t)} \frac{1}{h(\overline{s_1})}\int_{\phi(\alpha)}^{\phi(t_1)} f_2(\overline{s_2})\int_{\phi(\alpha)}^{\phi(t_2)} f_3(\overline{s_3})M^3|x(s_3)|g(|x(s_3)|)ds_3 ds_2 ds_1, \end{aligned} \tag{3.12}$$

where $s_i = t_i - \tau(t_i)$, $\overline{s_i} = s_i + \tau(t_i)$, $s_i \in I$ for $i = 1,2,3$. Now when $\varphi(u) = u$ and $f_1 = f_4 = \cdots = f_n = 0$, a suitable application of the inequality given in Theorem 2.6 to (3.12) yields the desired result. □

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Ravi P. Agarwal: Department of Mathematical Sciences, Florida Institute of Technology, 150 West University Boulevard, Melbourne, FL 32901-6975, USA
Email address: agarwal@fit.edu

Cheon Seoung Ryoo: Department of Mathematics, Hannam University, Daejeon 306-791, South Korea
Email address: ryoocs@hannam.ac.kr

Young-Ho Kim: Department of Applied Mathematics, Changwon National University, Changwon, Kyungnam 641-773, South Korea
Email address: yhkim@changwon.ac.kr